The reductions of the Darboux transformation and some solutions of the soliton equations

1996 J. Phys. A: Math. Gen. 29 4187

(http://iopscience.iop.org/0305-4470/29/14/036)

J. Phys. A: Math. Gen. **29** (1996) 4187–4195. Printed in the UK

The reductions of the Darboux transformation and some solutions of the soliton equations

Li Yi-Shen
Department of Mathematics, Center of Nonlinear Science (USTC), University of Science and Technology of China, Anhui, Hefei 230026, People's Republic of China

Received 18 January 1995, in final form 8 January 1996

Abstract. In this paper, we introduce a new technique into the old topic of Darboux transformations. In doing so, new solutions corresponding to the KP equation and a localized soliton solution of the DSIII equation are found explicitly.

1. Introduction

In [1] we considered the constraint of the Kadomtsev–Petviashvili (KP) equation

$$u_t = 3\partial_x^{-1} u_{yy} - u_{xxx} - 6uu_x \tag{1.1}$$

by the constraint

$$u(x, y, t) = -2 \sum_{j=1}^{m} p_j(x, y, t) q_j(x, y, t) \tag{1.2}$$

and reduction

$$p_j + q_j^* = 0 \qquad j = 1, 2, \ldots, m \tag{1.3}$$

where * denotes the complex conjugate and p_j satisfy the following equations:

$$\mathrm{i} p_{jy} = p_{jxx} + 2\left(\sum_{k=1}^{m} p_k p_k^* \right) p_j \tag{1.4}$$

$$p_{jt} = -4\left\{ p_{jxxx} + 3\left(\sum_{k=1}^{m} p_k p_k^* \right) p_{jx} + 3\left(\sum_{k=1}^{m} p_{kx} p_k^* \right) p_j \right\} \qquad j = 1, 2, \ldots, m\,. \tag{1.5}$$

If we solve this two $(1+1)$-dimensional evolution equation, then from (1.2) and (1.3), we get some solutions of the $(2+1)$-dimensional KP equation. Equations (1.4) and (1.5) are associated to the Lax pair (1.6), (1.7) and (1.6), (1.8), respectively,

$$\varphi_x = (\lambda J + M)\varphi \tag{1.6}$$

$$\mathrm{i}\varphi_y = (\lambda^2 J + \lambda M + M_1)\varphi \tag{1.7}$$

$$-\tfrac{1}{4}\varphi_t = (\lambda^3 J M + \lambda M_1 + M_2)\varphi \tag{1.8}$$

where λ is a parameter. φ is a $(m+1)$-component vector $\varphi = (\varphi_1, \ldots, \varphi_{m+1})$, J, M, M_1 are $(m+1) \times (m+1)$ matrices, and

$$J = \mathrm{diag}(1, 0, 0, \ldots, 0)$$

$$M = (M_i) \qquad M_{ii} = 0 \qquad M_{ij} = 0 \qquad i \neq j \qquad i \neq 1 \qquad j \neq 1$$

$$M_{1\,j+1} = p_j \qquad M_{i+1,1} = q_i \qquad i, j = 1, 2, \ldots, m$$

0305-4470/96/144187+09\$19.50

where $M_1 = \sigma(M_x - M^2)$, $M_2 = M_{xx} + MM_x - M_xM - 2M^3$ and $\sigma = \mathrm{diag}(1, -1, -1, \ldots, -1)$ as we know that the Darboux transformation (DT) is a powerful tool for finding the solution of the soliton equation, but we have to consider the DT under the reduction individually.

In section 2, we deal with the DT of the equations (1.6)–(1.8) under the reduction (1.3); the main technique is to introduce a bilinear form and rewrite the usual DT in an alternative form which is easy to use for computation. This is used to find some solutions of the KP equation in section 3. In section 4, we consider another (2 + 1)-dimensional equation [2],

$$\begin{aligned} &a_{xy} - a_{xx} + 2(bc)_x = 0 \qquad d_{xy} + d_{xx} - 2(bc)_x = 0 \\ &\mathrm{i}b_t + b_{xy} + 2b(d_x - a_x) = 0 \qquad -\mathrm{i}c_t + c_{xy} + 2c(d_x - a_x) = 0 \end{aligned} \tag{1.9}$$

where a_x, d_x are real functions and $b = c^*$, this equation is one member of the Davey–Stewartson (DS) hierarchy, and it is equivalent to the DSIII equation (we prove it in the appendix). By the constraint

$$w_x = pq^{\mathrm{T}} \qquad w = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad p^{\mathrm{T}} = (p_1, p_2) \qquad q^{\mathrm{T}} = (q_1, q_2) \tag{1.10}$$

and, reduction

$$p_j + q_j^* = 0 \tag{1.11}$$

where p, q satisfy the equations

$$p_y = \sigma_3 p_x + Qp \qquad q_y^{\mathrm{T}} = q^{\mathrm{T}}\sigma_3 - q^{\mathrm{T}}Q \qquad Q_x = [pq^{\mathrm{T}}, \sigma_3] \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.12}$$

$$-\mathrm{i}p_t = p_{xx} - 2w_x p \qquad \mathrm{i}q_t^{\mathrm{T}} = q_{xx}^{\mathrm{T}} - 2q^{\mathrm{T}}w_x\,. \tag{1.13}$$

If we can solve equations (1.12) and (1.13), then from $w_x - pq^{\mathrm{T}}$, one gets some solutions of equation (1.9).

The Lax pair for the equations (1.12) and (1.13) are (1.14), (1.15) and (1.14), (1.16), respectively.

$$f_x = Uf \qquad f^{\mathrm{T}} = (f_1, f_2, f_3) \qquad U = \begin{pmatrix} \xi & 0 & p_1 \\ 0 & \xi & p_2 \\ q_1 & q_2 & 0 \end{pmatrix} = \begin{pmatrix} \xi & p \\ q^{\mathrm{T}} & 0 \end{pmatrix} \tag{1.14}$$

$$f_y = U_1 f \qquad U_1 = \begin{pmatrix} \sigma_3\xi + Q & \sigma_3 p \\ q^{\mathrm{T}}\sigma_3 & 0 \end{pmatrix} \tag{1.15}$$

$$\mathrm{i}f_t = U_2 f \qquad U_2 = \begin{pmatrix} \xi^2 + pq^{\mathrm{T}} - 2Q_x & \xi p + p_x \\ \xi q^{\mathrm{T}} - q_x^{\mathrm{T}} & q^{\mathrm{T}}p \end{pmatrix} \tag{1.16}$$

where ξ is a spectral parameter.

We note that by the transformation $f_{j+1} = \psi_l \mathrm{e}^{\xi x}$ ($l \equiv j + 1, \mathrm{mod}\, 3$), $-\xi = \lambda$, the equation (1.14) is transformed to the type of the equation (1.6) with $m = 2$, such that one can get the DT for the equations (1.14)–(1.16) similarly, and we can use it to find the localized soliton solutions of the equation (1.9).

2. The DT of the spectral problem (1.6)

It is well known [4] that by the DT

$$\varphi' = (\lambda - S)\varphi \qquad S = H\Lambda H^{-1} \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{m+1}) \tag{2.1}$$

where H is a non-singular matrix solution of the equations (1.6)

$$H_x = JH\Lambda + MH\,. \tag{2.2}$$

Equation (2.1) maps (1.6) to the same types of equation

$$\varphi'_x = \lambda J\varphi' + M'\varphi' \tag{2.3}$$

and

$$M' = M + JS - SJ = M + [JS]\cdot S_x = M'S - SM\,. \tag{2.4}$$

The main problem is to select Λ and H such that the condition (1.3) $p'_j + q^*_j = 0$ is invariant after the DT (2.1).

Imposing the condition $\lambda_1 = \mu$, $\lambda_j = -p^*$, $j = 2, \ldots, m+1$, we denote by ψ, $\varphi^{(1)}$, $\varphi^{(j)}$ the solutions of (1.6) with spectral parameter λ, $\lambda = \mu$, and $\lambda = -\mu^*$, respectively, and we take

$$H = H_{ij} = \varphi^{(j)}\,. \tag{2.5}$$

Since

$$\varphi^{(1)}_x - (\mu J + M)\varphi^{(1)} \qquad \varphi^{(j)}_x - (-\mu^* + M)\varphi^{(j)} \tag{2.6}$$

by using the condition (1.3), from (2.6), it yields

$$((\varphi^{(j)*})^{\mathrm{T}}\varphi^{(1)})_x = (\varphi^{(j)*})^{\mathrm{T}}(M^+ + M)\varphi^{(1)} = 0 \tag{2.7}$$

where M^+ is a Hermite matrix of M.

This means that we can take $\varphi^{(j)}$ to satisfy the condition

$$(\varphi^{(j)}, \varphi^{(i)}) \equiv \sum_{l=1}^{m+1} \varphi^{(j)}_l \varphi^{(1)*}_l = 0 \qquad j = 2, 3, \ldots, n+1\,. \tag{2.8}$$

From equation (2.8), one gets the solution $\varphi^{(1)*}$,

$$\varphi^{(1)*}_i = \Delta^i_1 \tag{2.9}$$

where Δ^k_j is the cofactor of φ^k_i in the determinant and satisfies

$$\sum_{j=1}^{m+1} \varphi^j_i \Delta^k_j = \delta^k_i\,. \tag{2.10}$$

Since then,

$$\Delta \equiv \det H = \sum_{j=1}^{m+1} \varphi^{(1)}_j \varphi^{(1)*}_j = (\varphi^{(1)}, \varphi^{(1)})\,. \tag{2.11}$$

It is very important that the determinant is positive definite.

We introduce the bilinear form

$$[\psi, \varphi^{(1)}] = \frac{(\psi, \varphi^{(1)})}{\lambda + \mu^*} \qquad [\varphi^{(1)}, \varphi^{(1)}] = \frac{(\varphi^{(1)}, \varphi^{(1)})}{\mu + \mu^*}\,. \tag{2.12}$$

From equation (2.1), the matrix S can be expressed as follows:

$$S_{ij} = \left(\mu\varphi^{(1)}_i \Delta^j_1 - \mu^* \sum_{j=1}^{m+1} \varphi^{(k)}_j \Delta^{(j)}_k\right)\Big/\Delta\,. \tag{2.13}$$

By using (2.10), S_{ij} can be rewritten as follows:

$$S_{11} = -\mu^* + \frac{(\mu + \mu^*)\varphi^{(1)}_i \varphi^{(1)*}_i}{\Delta} \qquad S_{ij} = \frac{(\mu + \mu^*)\varphi^{(1)}_i \varphi^{(1)*}_j}{\Delta} \quad i \neq j\,. \tag{2.14}$$

We define

$$\psi_i\{1\} = \frac{\varphi_i'}{\lambda + \mu^*} \qquad i = 1, 2, \ldots, m+1\,. \tag{2.15}$$

By using (2.14), from (2.1) we have

$$\begin{aligned}\psi_i\{1\} &= \frac{1}{\lambda + \mu^*}\left\{\lambda\psi_i - \sum_{j=1}^{m+1} S_{ij}\psi_j\right\} \\ &= \psi_i + \frac{(\mu + \mu^*)\sum_{j=1}^{m+1}\varphi_j^{(1)}\varphi_j^{(1)*}\psi_j}{(\lambda + \mu^*)\Delta} \\ &= \frac{\begin{vmatrix} \psi_i & [\psi, \varphi^{(1)}] \\ \varphi_i^{(1)} & [\varphi^{(1)}, \varphi^{(1)}] \end{vmatrix}}{[\varphi^{(1)}, \varphi^{(1)}]}\,.\end{aligned} \tag{2.16}$$

We define

$$p_j\{1\} = p_j^t \qquad q_j\{1\} = q_j^t\,. \tag{2.17}$$

From equation (2.4), it yields

$$p_j\{1\} = p_j + S_{1j} = p_j + \frac{\varphi_1^{(1)}\varphi_j^{(1)*}}{[\varphi^{(1)}, \varphi^{(1)}]} \qquad q_j\{1\} = p_j - S_{j1} = q_1 - \frac{\varphi_j^{(1)}\varphi_1^{(1)*}}{[\varphi^{(1)}, \varphi^{(1)}]}\,. \tag{2.18}$$

It is easy to see that

$$p_j\{1\} + q_j^*\{1\} = 0\,.$$

The iteration of this DT gives

$$\psi\{N\} = \frac{\begin{vmatrix} \psi_i & [\psi, \varphi^1] & \cdots & [\psi, \varphi^N] \\ \varphi_i & [\varphi^1, \varphi^1] & \cdots & [\varphi^1, \varphi^N] \\ \vdots & \vdots & & \vdots \\ \varphi_i^N & [\varphi^N, \varphi^1] & \cdots & [\varphi^N, \varphi^N] \end{vmatrix}}{\Delta(N)} \tag{2.19}$$

$$p_j\{N\} = p_j + d_1^l\varphi_j^{l*} \qquad q_j\{N\} = q_j - d_j^l\varphi_1^{l*} \tag{2.20}$$

where φ^l is a solution of (1.6) with $\lambda = \mu l$, $[\psi, \varphi^l] = \frac{(\psi,\varphi^l)}{\lambda+\mu_l^*}$, $[\varphi^l, \varphi^k] = \frac{(\varphi^l,\varphi^l)}{\mu_i+\mu_i^*}$, $\Delta(N) = \det([\varphi^i, \varphi^j])$, $i, j = 1, \ldots, N$ and d_i^j are defined by expansion of the determinants.

3. Some solutions of the KP equation

The Lax pair (1.6)–(1.8), and consequently the compatibility condition, have the same Darboux covariance properties as (1.6) itself; equations (2.19) and (2.20) constitute the iterated DTs, where φ^j are the solution of Lax pair (1.6)–(1.8), we shall use it to generate the solutions of (1.4) and (1.5), which then give the solution of the KP equation from (1.2).

First of all we take $m = 1$. It is easy to see that $p_1 = -\mathrm{e}^{\mathrm{i}(x-y-20t)}$ is a solution of the equations (1.4) and (1.5), we put p_1 as our 'seed' into (1.6)–(1.8), and taking $\lambda = 2 + \mathrm{i}$ we find

$$\begin{aligned}\varphi_1' &= \left[e_1 - \frac{e_2}{12+24\mathrm{i}} + e_2 t - \frac{(1-\mathrm{i})e_2}{6+12\mathrm{i}}y - \frac{e_2 x}{12+24\mathrm{i}}\right]\mathrm{e}^{(1+\mathrm{i})x+2(1-\mathrm{i})y-(4+32\mathrm{i})t} \\ \varphi_2' &= \left[e_1 + e_2 t - \frac{(1-\mathrm{i})e_2}{6+12\mathrm{i}}y - \frac{e_2 x}{12+24\mathrm{i}}\right]\mathrm{e}^{x+(2-\mathrm{i})y-(4+12\mathrm{i})t}\end{aligned} \tag{3.1}$$

satisfy the equations (1.6)–(1.8) with the coefficients $p_1 = -e^{i(x-y-20t)}$, where e_1, e_2 are constants.

For simplicity, we take $e_1 = e_2 = 1$

$$\begin{aligned}\varphi_1' &= \left[\left(\tfrac{1}{36}+a\right) - i\left(\tfrac{1}{18}+b\right)\right]e^{(1+i)x+2(1-i)y-(4+32i)t}\\ \varphi_2' &= (a-bi)e^{x+(2-i)y-(4+12i)t}\end{aligned} \tag{3.2}$$

$$[\varphi', \varphi'] + \tfrac{1}{4}\left[a^2+b^2+\left(\tfrac{1}{36}+a\right)^2+\left(\tfrac{1}{18}+b\right)^2\right] = \tfrac{1}{4}\Delta \tag{3.3}$$

where

$$a = 1 + t - \tfrac{1}{18}y + \tfrac{1}{36}x \qquad b = \tfrac{1}{6}y + \tfrac{1}{18}x\,. \tag{3.4}$$

We get a new solution of the equation (1.4) and (1.5) with $m = 1$.

$$p_1\{1\} = \left(-1 + \frac{4\varphi_1'\varphi_2'^*}{\Delta}\right)e^{i(x-y-20t)}\,. \tag{3.5}$$

The new solution of the KP equation reads

$$u = -2p_1p_1^* = -2\left\{1 - \frac{8a\left(\frac{1}{36}+a\right)+8b\left(\frac{1}{18}+b\right)}{\Delta} + \frac{\left[\left(\frac{1}{36}+a\right)^2+\left(\frac{1}{18}+b\right)^2\right][a^2+b^2]}{\Delta^2}\right\} \tag{3.6}$$

where u is a rational function. Figure 1 shows $-\frac{1}{2}u(x, y, 0)$, when $t \to \infty$, $u \to 1$, which is different from the lump solutions.

Next we take $m = 2$, and $p_1 = e^{-2iy}$, $p_2 = 0$ as our 'seed' the solution of (1.6)–(1.8) reads

$$\begin{aligned}\varphi_1' &= e^{\delta+i\varepsilon-iy}\{e^{\lambda ay+i\nu}+e^{-\lambda ay-i\nu}\}\\ \varphi_2' &= -e^{\delta+i\theta+iy}\{e^{\lambda ay+i\nu-i\theta}+e^{-\lambda ay-i\nu+i\theta}\}\\ \varphi_3' &= 1\end{aligned} \tag{3.7}$$

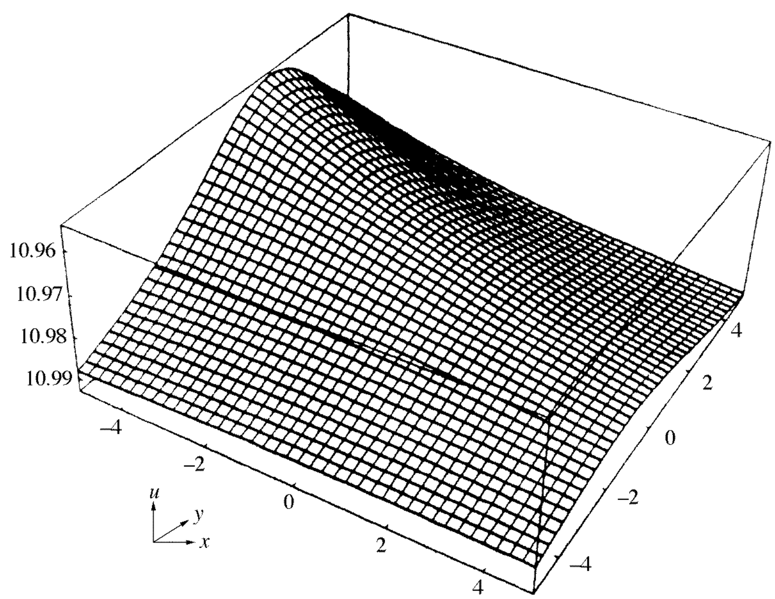

Figure 1.

where

$$\begin{aligned}
\varepsilon &= -\frac{\lambda^2}{2} y \\
\delta &= \frac{\lambda}{2} x + \lambda a y - 2\lambda^3 t \\
\nu &= a x - (4\lambda^2 a - 8a) t \\
a &= \sqrt{1 - \frac{\lambda^2}{4}} \qquad \lambda \text{ real and } |\lambda| < 2 \\
\theta &= \tan^{-1} \frac{\lambda}{2a} \\
[\varphi^1, \varphi^1] &= \frac{1}{\lambda} \{ e^{2\delta} [2\cosh 2\lambda a y + \cos 2\nu + \cos(2\nu + \theta)] + 1 \} \equiv \frac{1}{\lambda} \Delta \,.
\end{aligned} \tag{3.8}$$

We get the new solution of the equations (1.4) and (1.5) with $m = 2$,

$$\begin{aligned}
p_1\{1\} &= p_1 + \frac{\varphi_1^1 \varphi_2^{1*}}{[\varphi^1, \varphi^1]} \\
&= \left\{ 1 - \frac{2\lambda e^{2\delta} [\cos\theta \cosh 2\lambda a y - \mathrm{i} \sin\theta \sinh 2\lambda a y + \cos(2\nu + \theta)]}{\Delta} \right\} e^{-2\mathrm{i}y} \\
p_2\{1\} &= p_2 + \frac{\varphi_1^1 \varphi_3^{1*}}{[\varphi^1, \varphi^1]} \\
&= \frac{2\lambda e^{\delta + \mathrm{i}\epsilon - \mathrm{i}y} (\cos\nu \cosh 2\lambda a y - \mathrm{i} \sin\nu \sinh 2\lambda a y)}{\Delta}
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
u &= 2(|p_2\{1\}|^2 + |p_1\{1\}|^2) \\
&= 2 \left\{ \frac{4\lambda^2 e^{2\delta}}{\Delta^2} (\cos^2\nu \cosh^2 2\lambda a y + \sin^2\nu \sinh^2 2\lambda a y) + \frac{4\lambda^2 e^{4\delta} \sin^2\theta \sinh^2 2\lambda a y}{\Delta^2} \right. \\
&\qquad \left. + \left[1 - \frac{2\lambda e^{2\delta} \cos\theta \cosh 2\lambda a y + \cos(2\nu + \theta)]}{\Delta} \right]^2 \right\}
\end{aligned} \tag{3.10}$$

is a new solution of the KP equation to our knowledge.

4. The localized soliton solution of the equation (1.9)

The DT for (1.14)–(1.16) can be found similarly, for example, the solutions of the equation (1.12) and (1.13), $p_1[2]$, $p_2[2]$ can be expressed as follows:

$$\begin{aligned}
p_1\{2\} &= p_1 + \frac{\{\varphi_1^1 \varphi_3^{1*} [\varphi^2, \varphi^2] + \varphi_1^2 \varphi_3^{2*} [\varphi^1, \varphi^1] - \varphi_1^1 \varphi_3^{2*} [\varphi^2, \varphi^1] - \varphi_1^2 \varphi_3^{1*} [\varphi^1, \varphi^2]\}}{\Delta} \\
p_2\{2\} &= p_2 + \frac{\{\varphi_2^1 \varphi_3^{1*} [\varphi^2, \varphi^2] + \varphi_2^2 \varphi_3^{2*} [\varphi^1, \varphi^1] - \varphi_2^1 \varphi_3^{2*} [\varphi^2, \varphi^1] - \varphi_2^2 \varphi_3^{1*} [\varphi^1, \varphi^2]\}}{\Delta}
\end{aligned} \tag{4.1}$$

$$\Delta = [\varphi^1, \varphi^1][\varphi^2, \varphi^2] - [\varphi^1, \varphi^2][\varphi^2, \varphi^1] \tag{4.2}$$

where φ^1, φ^2 are solutions of the equation (1.14)–(1.16) with $\xi = \mu_1$, $\xi = \mu_2$, respectively.

To solve the equation

$$w_x\{2\} = p\{2\} q^{\mathrm{T}}\{2\} \qquad q^*\{2\} + p\{2\} = 0 \tag{4.3}$$

one gets the solution of the equation (1.9), but we can get it in an alternative way.

We write the S matrix as follows:

$$S = \begin{pmatrix} T & E \\ F & h \end{pmatrix} \tag{4.4}$$

where T is a 2×2 matrix, E and F are 1×2 and 2×1 matrices, respectively, and h is a 1×1 matrix. Then

$$p\{1\} = p + E \qquad q^{\mathrm{T}}\{1\} = q^{\mathrm{T}} - F\,. \tag{4.5}$$

Now we are interested in finding the solution w of the equation (1.9). After the DT

$$(w\{1\})_x = p\{1\}_q^{\mathrm{T}}\{1\} = w - pF - EF + Eq^{\mathrm{T}}\,. \tag{4.6}$$

From equation (2.4)

$$S_x = \begin{pmatrix} T & E \\ F & h \end{pmatrix}_x = \begin{pmatrix} 0 & p+E \\ q^* - F & 0 \end{pmatrix} \begin{pmatrix} T & E \\ F & h \end{pmatrix} - \begin{pmatrix} T & E \\ F & h \end{pmatrix} \begin{pmatrix} 0 & p \\ q^{\mathrm{T}} & 0 \end{pmatrix}$$

it yields

$$T_x = pF + EF - E_q^{\mathrm{T}}\,. \tag{4.7}$$

Combining (4.3) and (4.4), we have

$$w\{1\} = w - T$$

or

$$a\{1\} = a + \mu_1^* - \frac{|\varphi_1^1|^2}{[\varphi^1, \varphi^1]} \qquad b\{1\} = c^*\{1\} = b - \frac{\varphi_1^1 \varphi_2^{1*}}{[\varphi^1, \varphi^1]}$$
$$d\{1\} = d + \mu^* - \frac{|\varphi_2^1|^2}{[\varphi^1, \varphi^1]}\,. \tag{4.8}$$

By iterations we get

$$a\{2\} = a + \mu_1^* + \mu_2^* - \frac{\{|\varphi_1^1|^2[\varphi^2, \varphi^2] + |\varphi_1^2|^2[\varphi^1, \varphi^1] - \varphi_1^1\varphi_1^{2*}[\varphi^2, \varphi^1] - \varphi_1^2\varphi_1^{1*}[\varphi^1, \varphi^2]\}}{\Delta}$$
$$d\{2\} = d + \mu_1^* + \mu_2^* - \frac{\{|\varphi_2^1|^2[\varphi^2, \varphi^2] + |\varphi_2^2|^2[\varphi^1, \varphi^1] - \varphi_2^1\varphi_2^{2*}[\varphi^2, \varphi^1] - \varphi_2^2\varphi_2^{1*}[\varphi^1, \varphi^2]\}}{\Delta}$$
$$c^*\{2\} = b\{2\} = b - \frac{\{\varphi_1^1\varphi_2^{1*}[\varphi^2, \varphi^2] + \varphi_1^2\varphi_2^{2*}[\varphi^1, \varphi^1] - \varphi_1^1\varphi_2^{2*}[\varphi^2, \varphi^1] - \varphi_1^2\varphi_2^{1*}[\varphi^1, \varphi^2]\}}{\Delta} \tag{4.9}$$

$$\Delta = [\varphi^1, \varphi^1][\varphi^2, \varphi^2] - [\varphi^1, \varphi^2][\varphi^2, \varphi^1]\,. \tag{4.10}$$

Now we begin with the 'seed' $p^{\mathrm{T}} = (0, 0)$, $q^{\mathrm{T}} = (0, 0)$, the solution φ^1, φ^2 of the equation (1.14)–(1.16) reads

$$\varphi_1^1 = c_1 \mathrm{e}^{\mu_1(x+y) + \mathrm{i}\mu_1^2 t} \qquad \varphi_1^2 = d_1 \mathrm{e}^{\mu_2(x+y) + \mathrm{i}\mu_2^2 t}$$
$$\varphi_2^1 = c_2 \mathrm{e}^{\mu_1(x-y) + \mathrm{i}\mu_1^2 t} \qquad \varphi_2^2 = d_2 \mathrm{e}^{\mu_2(x-y) + \mathrm{i}\mu_2^2 t} \tag{4.11}$$
$$\varphi_3^1 = c_3 \qquad \varphi_3^2 = d_3$$

where $\mu_1 = \xi_1 + \mathrm{i}\xi_2$, $\mu_2 = \eta_1 + \mathrm{i}\eta_2$. For simplicity, we take c_j, d_j, $(j = 1, 2, 3)$ to be real, then

$$
\begin{aligned}
[\varphi^1, \varphi^1] &= \frac{1}{2\xi_1}(c_1^2 \mathrm{e}^{2\delta_1} + c_2^2 \mathrm{e}^{2\delta_2} + c_3^2) \\
[\varphi^2, \varphi^2] &= \frac{1}{2\eta_1}(d_1^2 \mathrm{e}^{2\rho_1} + d_2^2 \mathrm{e}^{2\rho_2} + d_3^2) \\
[\varphi^2, \varphi^1]^* &= [\varphi^1, \varphi^2] \\
&= \frac{1}{(\xi_1 + \eta_1) + (\xi_2 - \eta_2)\mathrm{i}}(c_1 d_1 \mathrm{e}^{\delta_1 + \rho_1} \mathrm{e}^{\mathrm{i}(\varepsilon_1 - \nu_1)} + c_2 d_2 \mathrm{e}^{\delta_2 + \rho_2} \mathrm{e}^{\mathrm{i}(\varepsilon_2 - \nu_2)} + c_3 d_3)
\end{aligned}
\tag{4.12}
$$

where

$$
\begin{aligned}
&\delta_1 = \xi_1(x + y) - 2\xi_1\xi_2 t \qquad \delta_2 = \xi_1(x - y) - 2\xi_1\xi_2 t \\
&\varepsilon_1 = \xi_2(x + y) + (\xi_1^2 - \xi_2^2)t \qquad \varepsilon_2 = \xi_2(x - y) + (\xi_1^2 - \xi_2^2)t \\
&\rho_1 = \eta_1(x - y) - 2\eta_1\eta_2 t \qquad \rho_2 = \eta_1(x - y) - 2\eta_1\eta_2 t \\
&\nu_1 = \eta_2(x + y) + (\eta_1^2 - \eta_2^2)t \qquad \nu_2 = \eta_2(x - y) + (\eta_1^2 - \eta_2^2)t\,.
\end{aligned}
\tag{4.13}
$$

Substituting (4.9), (4.10) into (4.6), we get the solution of equation (1.9). Now we consider a special case

$$
e_2 = d_1 = 0 \qquad c_3 = d_3 = \sqrt{\tfrac{3}{2}} \qquad c_1 = d_2 = \sqrt{\tfrac{1}{2}} \qquad \xi_1 = 2 \qquad \xi_2 = \eta_1 = \eta_2 = 1 \tag{4.14}
$$

$$
\begin{aligned}
b\{2\} &= 4\mathrm{e}^{\mathrm{i}(2y+3t)}/[\cosh(3x + y - 6t) + 3\cosh(x + 3y - 2t)] \\
a\{2\} &= 2 - \mathrm{i} - 4\cosh(3x_u - 6t)/[\cosh(3x + y - 6t) + 3\cosh(x + 3y - 2t)]
\end{aligned}
\tag{4.15}
$$

where $b\{2\}$ is a localized soliton solution (figure 2 shows $\frac{1}{4}|b\{2\}|$ with $t = 0$). This localized soliton may be similar to the solution in [3].

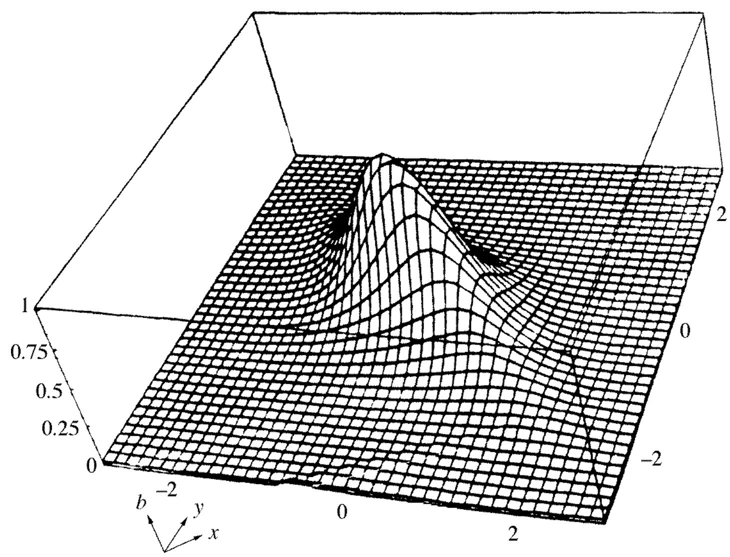

Figure 2.

Acknowledgments

This work was supported by the Chinese National Basic Research Project 'Nonlinear Science' and the Ministry of Education of China. The author thanks the referees for suggestions.

Appendix

In [3], the authors deduced the DSIII equations as follows:

$$\begin{aligned} &\mathrm{i}q_t - q_{uu} + q_{vv} + (-\varphi_{uu} + \varphi_{vv})q = 0 \\ &-\mathrm{i}r_t - r_{uu} + r_{vv} + (-\varphi_{uu} + \varphi_{vv})q = 0 \\ &2\varphi_{uv} = -qr\,. \end{aligned} \tag{A.1}$$

By the transformation $x = (u + v)/2$, $y = (v - u)/2$, equation (A.1) becomes

$$\begin{aligned} &\mathrm{i}q_t + q_{xy} + \varphi_{xy} q = 0 \\ &-\mathrm{i}r_t + r_{xy} + \varphi_{xy} r = 0 \\ &\tfrac{1}{2}(\varphi_{xx} - \varphi_{yy}) = -qr\,. \end{aligned} \tag{A.2}$$

Now we assume that the function φ_{xy} is integrated with respect to x or y and the function a_{xy}, d_{xy} are integrated with respect to x, and all these constants of integration are chosen to be zero,

$$d_x - a_x = -(a_y + d_y) \qquad a_y - a_x - d_y - d_x = -4bc\,. \tag{A.3}$$

Comparing the last two equations of (1.9) with the first two equations of (A.2), we take

$$\varphi_{xy} = 2(d_x - a_x) \qquad \text{or} \qquad \varphi_y = -2(d - a) \qquad \varphi_{yy} = 2(d_y - a_y)\,. \tag{A.4}$$

Using equation (A.3), we have

$$\varphi_{xy} = -2(d_y + a_y) \qquad \text{or} \qquad \varphi_x = -2(d + a) \qquad \varphi_{xx} = -2(d_x + a_x) \tag{A.5}$$

which yields

$$\tfrac{1}{2}(\varphi_{xx} - \varphi_{yy}) = (-d_x - a_x - d_y + a_y) = -4bc\,. \tag{A.6}$$

By using (A.4) and (A.6), we let $b = q/2$, $c = 1/2$; equation (1.9) is reduced to (A.2).

References

[1] Chen Yi and Li Yi-Shen 1991 *Phys. Lett.* **157A** 22
[2] Li Yi-Shen 1993 *J. China USTC* **23** 1
[3] Pempinali F 1992 Localized soliton solution for DSI and DSIII equation *Preprint*
[4] Matveev V B and Salle M 1991 *Darboux Transformation and Soliton* (Berlin: Springer)